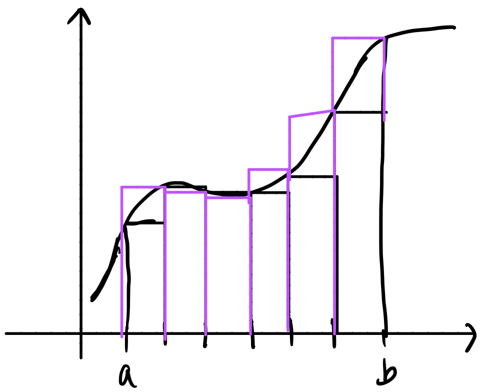
LECTURE: 5-1 AREAS AND DISTANCES

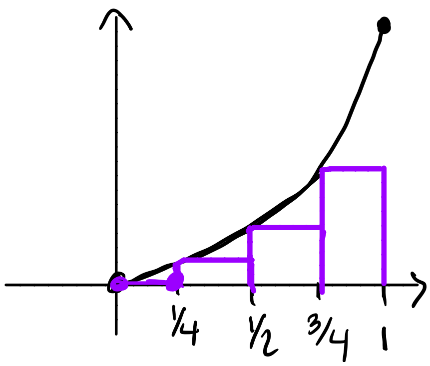
Areas - The Big Question: How Might you Find Area Under a Curvy Curve?



1. divide $[a,b]$ into n pieces (subintervals)
2. estimate area of each sub-interval w/ a rectangle.
3. make it better? Make more rectangles!

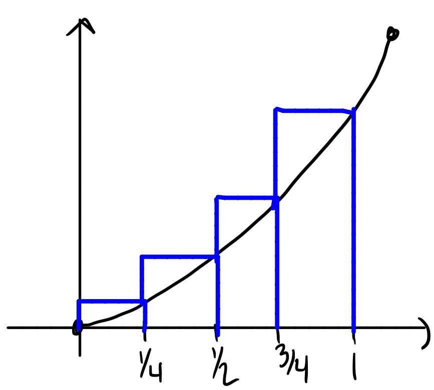
Example 1: Divide the interval $[0, 1]$ into $n = 4$ sub-intervals of equal width. Then, use four rectangles to estimate the area under $y = x^2$ from 0 to 1.

(a) Using left endpoints. Width of sub-intervals is $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = 1/4$



$$L_4 = \frac{1}{4}(0) + \frac{1}{4}f(1/4) + \frac{1}{4}f(1/2) + \frac{1}{4}f(3/4)$$
$$= \frac{1}{4}\left(0^2 + (1/4)^2 + (1/2)^2 + (3/4)^2\right)$$
$$= \frac{1}{4}\left(\frac{1}{16} + \frac{1}{4}\frac{4}{4} + \frac{9}{16}\right)$$
$$= \frac{1}{4}\cdot\frac{14}{16}$$
$$= \boxed{7/32 \approx 0.21875}$$

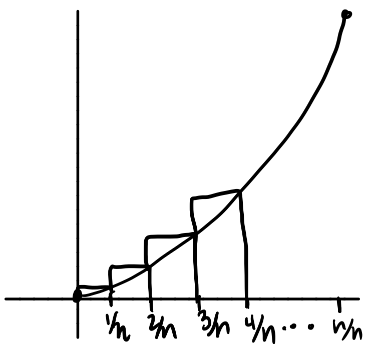
(b) Using right endpoints.



$$R_4 = \frac{1}{4}f(1/4) + \frac{1}{4}f(1/2) + \frac{1}{4}f(3/4) + \frac{1}{4}f(1)$$
$$= \frac{1}{4}\left((1/4)^2 + (1/2)^2 + (3/4)^2 + 1^2\right)$$
$$= \frac{1}{4}\left(1/16 + 1/4\,\frac{4}{4} + 9/16 + \frac{1}{1}\,\frac{16}{16}\right)$$
$$= \frac{1}{4}\left(\frac{30}{16}\right)$$
$$= \boxed{15/32 \approx 0.46875}$$

To find the actual area we need to take the number of sub-intervals to infinity. To do this we need a general expression for the left or right estimate for any n. This process is rather tedious and we will soon learn how we can use Calculus to find area under curves without having to use this long, tedious process.

Example 2: Prove that the area under $y = x^2$ from 0 to 1 is $\frac{1}{3}$.



$$R_n = \frac{1}{n}\left(f(1/n) + f(2/n) + f(3/n) + \dots + f(n/n)\right)$$

$$= \frac{1}{n}\left(\frac{1^2}{n^2} + \frac{2^2}{n^2} + \frac{3^2}{n^2} + \dots + \frac{n^2}{n^2}\right)$$

$$= \frac{1}{n^3}\left(\underbrace{1^2 + 2^2 + 3^2 + \dots + n^2}\right)$$

this adds up to $\frac{n(n+1)(2n+1)}{6}$

$$= \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6n^2}$$

$$= \frac{2n^2 + n + 2n + 1}{6n^2}$$

$$= \frac{2n^2 + 3n + 1}{6n^2}$$

this gives the right sum for any n

To get the exact area we take n (# of sub-intervals) to ∞.

$$A = \lim_{n\to\infty} R_n$$

$$= \lim_{n\to\infty} \frac{(2n^2 + 3n + 1)\,1/n^2}{(6n^2)\,1/n^2}$$

$$= \lim_{n\to\infty} \frac{(2 + 3/n + 1/n^2)}{6}$$

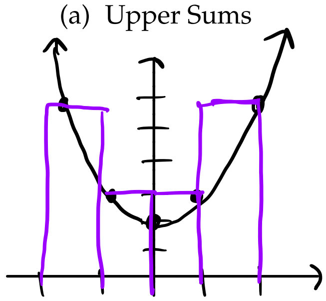
$$= \boxed{1/3}$$

← area under $y = x^2$ on $[0,1]$ is exactly $1/3$.

> **Upper and Lower Sums:** In general, we form **lower** (and **upper**) **sums** by choosing the sample points x_i^* so that $f(x_i^*)$ is the minimum (and maximum) value of f on the ith sub-interval.

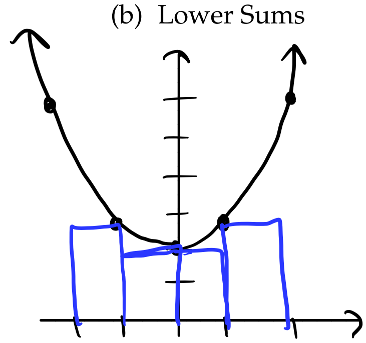
Example 3: Estimate the area under $f(x) = 2 + x^2$, $[-2, 2]$ with $n = 4$ using $\Delta x = \frac{2-(-2)}{4} = 1$

(a) Upper Sums



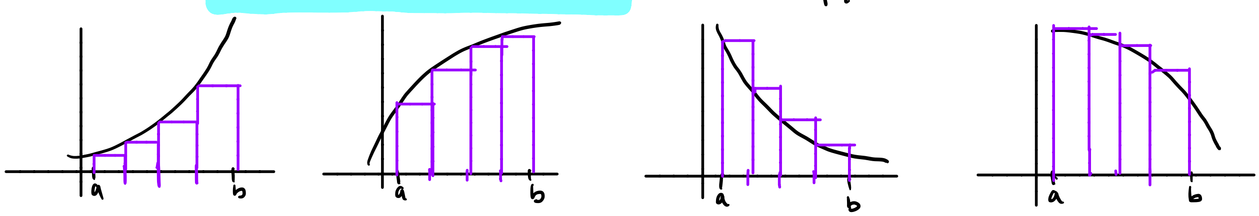
$U_4 = 1\left(f(-2) + f(-1) + f(1) + f(2)\right)$
$= (6 + 3 + 3 + 6)$
$= 18$

(b) Lower Sums



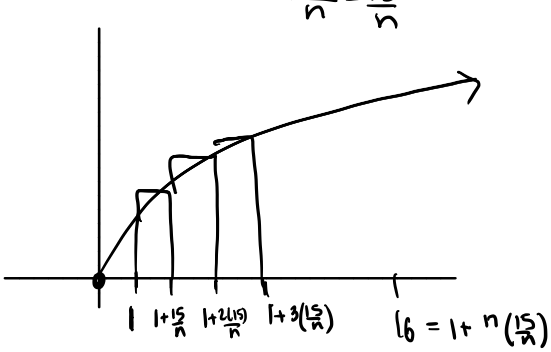
$\text{Lower} = 1\left(f(-1) + f(0) + f(0) + f(1)\right)$
$= 3 + 2 + 2 + 3$
$= 10$

Question: What type of behavior will guarantee that the left sum is an under-estimate and the right sum is an over-estimate? increase / decrease? concave up / concave down?



If f is increasing L_n is under-est, R_n is over-est
If f is decreasing L_n is over-est, R_n is under-est

Example 4: Find an expression for the area under the graph of $f(x) = \sqrt{x}$, $1 \le x \le 16$ as a limit. Do NOT evaluate the limit. $\Delta x = \frac{16-1}{n} = \frac{15}{n}$



$R_n = \frac{15}{n} \cdot \left(\sqrt{1 + 1\cdot\frac{15}{n}} + \sqrt{1 + 2\cdot\frac{15}{n}} + \sqrt{1 + 3\cdot\frac{15}{n}} + \cdots \sqrt{1 + n\cdot\frac{15}{n}}\right)$

$= \sum_{i=1}^{n} \frac{15}{n}\sqrt{1 + i\left(\frac{15}{n}\right)}$

$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{15}{n}\sqrt{1 + i\left(\frac{15}{n}\right)}$

Example 5: Determine a region whose area is equal to the given limit.

$\Delta x =$ width of interval

starting point.

(a) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(5+\frac{2i}{n}\right)^{10}$

This is the area under $f(x) = x^{10}$ on $[5, 7]$.

$\Delta x =$ width of interval

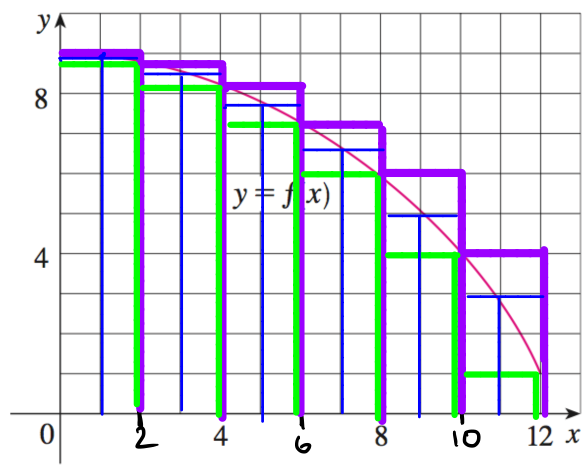
starting point

(b) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{5}{n}\sin\left(2+\frac{5i}{n}\right)$

This is the area under $f(x) = \sin x$ on $[2, 7]$

Example 6:

(a) Use six rectangles to find estimates of each type for the area under the given graph of f from $x = 0$ to $x = 12$.



(i) $L_6 \approx 2(9+8.8+8.2+7.2+6+4)$

$= \boxed{86.4}$

(ii) $R_6 \approx 2(8.8+8.2+7.2+6+4+1)$

$= \boxed{70.4}$

(iii) $M_6 \approx 2(8.9+8.5+7.7+6.5+5+3)$

$= \boxed{79.2}$

(b) Is L_6 an underestimate or overestimate of the true area? Is R_6 an underestimate or overestimate of the true area?

L_6 is an overestimate

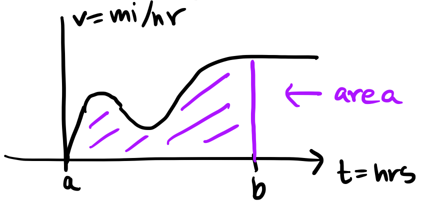
R_6 is an underestimate

(c) Which of the numbers L_6, R_6 or M_6 gives the best estimate? Explain.

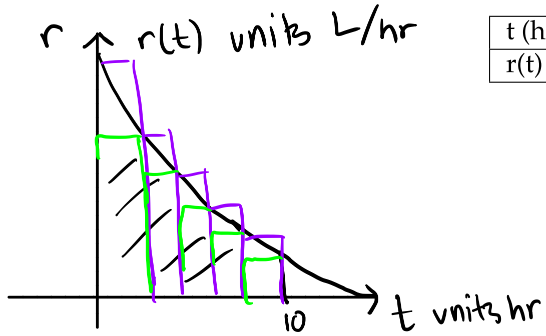
M_6 appears to be best as it's over-estimates seem to cancel out w/ underestimates.

Distances

If velocity is constant dist = vel * time



Example 7: Oil leaked out of a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at 2 hour time intervals are shown in the table. Find lower and upper estimates for the total amount of oil that leaked out.



t (h)	0	2	4	6	8	10
r(t) (L/h)	8.7	7.6	6.8	6.2	5.7	5.3

upper est is L_5

$L_5 = 2(8.7 + 7.6 + 6.8 + 6.2 + 5.7)$

$= \boxed{70\ L}$

lower est. is R_5

$R_5 = 2(7.6 + 6.8 + 6.2 + 5.7 + 5.3)$

$= \boxed{63.2\ L}$

A has units $L/hr \cdot hr$ = Liters!

Example 8: Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30 second time interval. We take speedometer readings every five seconds and then record them in the table below. Estimate the distance traveled by the car using a left sum and a right sum.

warning! not the same units

convert mi/hr → ft/sec

250+14

$\frac{1\ mi}{1\ hr} \cdot \frac{5280\ ft}{1\ mi} \cdot \frac{1\ hr}{60\ min} \cdot \frac{1\ min}{60\ sec}$

$= \frac{5280}{3600} = \frac{528}{360} = \frac{264}{180} = \frac{132}{90} = \frac{66}{45}$ *

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28
ft/sec	24.933	30.8	35.2	42.533	46.933	45.467	41.067

$L_6 = 5(24.933 + 30.8 + 35.2 + 42.533 + 46.933 + 45.467)$

$= \boxed{1,129.33\ ft}$

$R_6 = 5(30.8 + 35.2 + 42.533 + 46.933 + 45.467 + 41.067)$

$= \boxed{1210\ ft}$